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Fermi motion parameter $p_{_F}$ of B meson from relativistic quark model

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ABSTRACT

The Fermi motion parameter $p_{_F}$ is the most important parameter of ACCMM model, and the value $p_{_F} \sim 0.3$ GeV has been used without clear theoretical or experimental evidence. So, we attempted to extract the possible value for $p_{_F}$ theoretically in the relativistic quark model using quantum mechanical variational method. We obtained $p_{_F} \sim 0.5$ GeV, which is somewhat larger than 0.3, and we also derived the eigenvalue of $E_B \simeq 5.5$ GeV, which is is reasonable agreement with $m_B = 5.28$ GeV.

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In the minimal standard model CP violation is possible through the CKM mixing matrix of three families, and it is important to know whether the element V_{ub} is non-zero or not accurately. Its knowledge is also necessary to check whether the unitarity triangle is closed or not [1]. However its experimental value is very poorly known until now and its better experimental information is urgently required. At present, the only experimental method to study V_{ub} has been through the end-point leptonic spectrum of the B-meson semileptonic decays of CLEO [2] and ARGUS [3], and their data indicate that V_{ub} is non-zero. Recntly it has also been suggested [4] that the measurement of hadronic invariant mass distribution in the inclusive $B \to X_{c(u)} l\nu$ decays can be useful in extracting $|V_{ub}|$ with better theoretical understandings. In future asymmetric B factories with vertex detector, this will offer an alternative way to select $b \to u$ transitions that is much more efficient than selecting the upper end of the lepton energy spectrum.

The simplest model for the semileptonic B-decay is the spectator model which considers the decaying b-quark in the B-meson as a free particle. The spectator model is usually used with the inclusion of perturbative QCD radiative corrections. The decay width of the process $B \to X_q l\nu$ is given by

$$\Gamma_B(B \to X_q l\nu) \simeq \Gamma_b(b \to q l\nu) = |V_{bq}|^2 (\frac{G_F^2 m_b^5}{192\pi^3}) f(\frac{m_q}{m_b})[1 - \frac{2}{3}\frac{\alpha_s}{\pi} g(\frac{m_q}{m_b})] \quad , \qquad (1)$$

where m_q is the mass of the q-quark decayed from b-quark. The decay width of the spectator model depends on m_b^5, therefore small difference of m_b would change the decay width significantly.

Altarelli et al. [5] proposed their ACCMM model for the inclusive B-meson semileptonic decays. This model incorporates the bound state effect by treating the b-quark as a vitual state particle, thus giving momentum dependence to the b-quark mass. The virtual state b-quark mass W is given by

$$W^2(\mathbf{p}) = m_B^2 + m_{sp}^2 - 2m_B\sqrt{\mathbf{p}^2 + m_{sp}^2} \qquad (2)$$

in the B-meson rest frame, where m_{sp} is the spectator quark mass, m_B the B-meson

mass, and $\mathbf{p}$ is the momentum of the b-quark.

For the momentum distribution of the virtual b-quark, Altarelli et al. considered the Fermi motion inside the B-meson as the Gaussian distribution

$$\phi(\mathbf{p}) = \frac{4}{\sqrt{\pi} p_{_F}^3} e^{-\mathbf{p}^2/p_{_F}^2} \tag{3}$$

with a free parameter $p_{_F}$ of Gaussian width. And the decay width is given by integrating the width Γ_b in (1) with the weight $\phi(\mathbf{p})$. Then the leptonic spectrum of the B-meson semileptonic decay is given by

$$\frac{d\Gamma_B}{dE_l}(p_{_F}, m_{sp}, m_q, m_B) = \int_0^{p_{max}} dp\; p^2 \phi(\mathbf{p})\; \frac{d\Gamma_b}{dE_l}(m_b = W, m_q) \quad , \tag{4}$$

where p_{max} is the maximum kinematically allowed value of $p = |\mathbf{p}|$. The ACCMM model, therefore, introduces a new parameter $p_{_F}$ for the momentum measure of the Gaussian distribution inside B-meson instead of the b-quark mass of the spectator model. In this way the ACCMM model incorporates the bound state effects and reduces the strong dependence on b-quark mass in the decay width of the spectator model.

The Fermi motion parameter $p_{_F}$ is the most essential parameter of the ACCMM model as explained above. However, the experimental determinations of its value from the leptonic spectrum have been very ambiguous until now because the various effects from the input parameters and the perturbative QCD corrections are intermixed in the spectrum. The value $p_{_F} \sim 0.3$ has been widely used in experimental analyses without theoretical or experimental clean justification, even though there has been recently an assertion that the BSUV model [6] is approximately equal to ACCMM model at $p_{_F} \simeq 0.3$. Therefore, it is strongly required to determine the value of $p_{_F}$ more firmly when we think of the importance of its role in experimental analyses. The better determination of $p_{_F}$ is also interesting theoretically since it has the physical correspondence related to the Fermi motion inside B-meson. In this context we are going to theoretically determine the value of $p_{_F}$ in the relativistic quark model using quantum mechanical variational method.

We consider the Gaussian probability distribution function $\phi(\mathbf{p})$ in (3) as the absolute square of the momentum space wave function $\chi(\mathbf{p})$ of the bound state B-meson, i.e.,

$$\phi(\mathbf{p}) = 4\pi|\chi(\mathbf{p})|^2 \quad , \tag{5}$$

$$\chi(\mathbf{p}) = \frac{1}{(\sqrt{\pi}p_{_F})^{3/2}} e^{-\mathbf{p}^2/2p_{_F}^2} \quad . \tag{6}$$

The Fourier transform of $\chi(\mathbf{p})$ gives the coordinate space wave function $\psi(\mathbf{r})$, which is also Gaussian,

$$\psi(\mathbf{r}) = (\frac{p_{_F}}{\sqrt{\pi}})^{3/2} e^{-r^2 p_{_F}^2/2} \quad . \tag{7}$$

Then we can approach the determination of $p_{_F}$ in the framework of quantum mechanics. We apply the variational method with the Hamiltonian operator

$$H = \sqrt{\mathbf{p}^2 + m_{sp}^2} + \sqrt{\mathbf{p}^2 + m_b^2} + V(r) \tag{8}$$

and the trial wave function

$$\psi(\mathbf{r}) = (\frac{\mu}{\sqrt{\pi}})^{3/2} e^{-\mu^2 r^2/2} \quad , \tag{9}$$

where μ is the variational parameter. The ground state is given by minimizing the expectation value of H,

$$\langle H \rangle = \langle \psi | H | \psi \rangle = E(\mu) \quad , \tag{10}$$

$$\frac{d}{d\mu} E(\mu) = 0 \quad \text{at} \quad \mu = \bar{\mu} \quad , \tag{11}$$

and then $\bar{\mu} = p_{_F}$ and $\bar{E} \equiv E(\bar{\mu})$ approximates m_B. The value of μ or $p_{_F}$ corresponds to the measure of the radius of the two body bound state as can be seen

from

$$\langle r \rangle = \frac{2}{\sqrt{\pi}} \frac{1}{\mu} \quad , \quad \text{and} \quad \langle r^2 \rangle^{\frac{1}{2}} = \frac{3}{2} \frac{1}{\mu} \quad . \tag{12}$$

In (8), for simplicity, we take the Cornell potential which is composed of the coulomb and linear potentials,

$$V(r) = -\frac{\alpha_c}{r} + Kr \quad . \tag{13}$$

For the values of the parameters α_c ($\equiv \frac{4}{3}\alpha_s$), K, and the b-quark mass m_b, we use the values given by Hagiwara et al. [7],

$$\alpha_c = 0.47 \ (\alpha_s = 0.35), \quad K = 0.19 \ GeV^2, \quad m_b = 4.75 \ GeV, \tag{14}$$

which have been determined by the best fit of the $(c\bar{c})$ and $(b\bar{b})$ bound states. We will also consider the following value of α_c for comparison in our analysis

$$\alpha_c = 0.32 \ (\alpha_s = 0.24) \quad , \tag{15}$$

which corresponds to $\alpha_s(Q^2 = m_B^2)$.

Before applying our variational method with the Gaussian trial wave function to the B-meson system, let us check the method by considering the $\Upsilon(b\bar{b})$ system. The Hamiltonian of the $\Upsilon(b\bar{b})$ system can be approximated by the non-relativistic Hamiltonian

$$H \simeq 2m_b + \frac{\mathbf{p}^2}{m_b} + V(r) \quad . \tag{16}$$

With the parameters in (14) or (15), our variational method with the Gaussian trial wave function gives $p_{_F} = \bar{\mu} = 1.1 \ GeV$ and $\bar{E} = E(\bar{\mu}) = 9.49 \ GeV$. Here $p_{_F} = 1.1 \ GeV$ corresponds to the radius $R(\Upsilon) = 0.2 \ fm$, and $\bar{E}(\Upsilon) = 9.49 \ GeV$ is within 0.3 % error compared with the experimental value $E_{\rm exp} = m_\Upsilon = 9.46 \ GeV$. Therefore, the variational method with the non-relativistic Hamiltonian gives the fairly accurate results for the Υ ground state.

However, since the u- or d- quark in the B-meson is very light, the non-relativistic description can not be applied to the B-meson system. For example, when we apply the variational method with the non-relativistic Hamiltonian to the B-meson, we get the following results

$$p_{_F} = 0.29\ GeV, \quad \bar{E} = 5.92\ GeV \ \text{ for } \ \alpha_s = 0.35, \tag{17}$$

$$p_{_F} = 0.29\ GeV, \quad \bar{E} = 5.97\ GeV \ \text{ for } \ \alpha_s = 0.24. \tag{18}$$

The above masses $\bar{E}$ are much larger compared to the experimental value $m_B = 5.28\ GeV$, and moreover the expectation values of the higher terms in the non-relativistic perturbative expansion are bigger than those of the lower terms. Therefore, we can not apply the variational method with the non-relativistic Hamiltonian to the B-meson system.

We use the following Hamiltonian for the B-meson system in our analysis by treating u- or d-quark relativistically,

$$H = M + \frac{\mathbf{p}^2}{2M} + \sqrt{\mathbf{p}^2 + m^2} + V(r) \quad , \tag{19}$$

where $M = m_b$ and $m = m_{sp}$.

In our variational method the trial wave function is Gaussian both in the coordinate space and in the momentum space, so the expectation value of H can be calculated in either space,

$$\langle H \rangle = \langle \psi(\mathbf{r}) | H | \psi(\mathbf{r}) \rangle = \langle \chi(\mathbf{p}) | H | \chi(\mathbf{p}) \rangle \quad . \tag{20}$$

Also, the Gaussian function is a smooth function and its derivative of any order is square integrable, thus any power of the Laplacian operator ∇^2 is a hermitian operator at least under Gaussian functions. Therefore, analyzing the Hamiltonian (19) with the variational method can be considered as reasonable even though solving the eigenvalue equation of the differential operator (19) may be confronted with the mathematical difficulties because of the square root operator in (19).

With the Gaussian trial wave function (6) or (9), the expectation value of H can easily be calculated besides the square root operator,

$$\langle \mathbf{p}^{\,2} \rangle = \langle \psi(\mathbf{r}\,) | \mathbf{p}^{\,2} | \psi(\mathbf{r}\,) \rangle = \langle \chi(\mathbf{p}\,) | \mathbf{p}^{\,2} | \chi(\mathbf{p}\,) \rangle = \frac{3}{2}\mu^2 \quad , \tag{21}$$

$$\langle V(r) \rangle = \langle \psi(\mathbf{r}) | -\frac{\alpha_c}{r} + Kr | \psi(\mathbf{r}) \rangle = \frac{2}{\sqrt{\pi}}(-\alpha_c\mu + K/\mu) \quad . \tag{22}$$

Now let us consider the expectation value of the square root operator in the momentum space

$$\begin{aligned} \langle \sqrt{\mathbf{p}^{\,2} + m^2} \rangle &= \langle \chi(\mathbf{p}\,) | \sqrt{\mathbf{p}^{\,2} + m^2} | \chi(\mathbf{p}\,) \rangle \\ &= \Big(\frac{\mu}{\sqrt{\pi}}\Big)^3 \int_0^\infty e^{-p^2/\mu^2} \sqrt{\mathbf{p}^{\,2} + m^2}\; d^3p \\ &= \frac{4\mu}{\sqrt{\pi}} \int_0^\infty e^{-x^2} \sqrt{x^2 + (m/\mu)^2}\; x^2 dx \quad . \end{aligned} \tag{23}$$

The integral (23) can be given as a series expansion by the following procedure. First, define

$$I(s) \;\equiv \int_0^\infty \sqrt{x^2+s}\; x^2 e^{-x^2} dx = s^2 \int_0^\infty \sqrt{t^2+1}\; t^2 e^{-st^2} dt \quad , \tag{24}$$

$$I_0(s) \equiv \int_0^\infty \sqrt{x^2+s}\; e^{-x^2} dx = s \int_0^\infty \sqrt{t^2+1}\; e^{-st^2} dt \quad . \tag{25}$$

Next, from (24) and (25), we find the following differential relations

$$\frac{d}{ds}\Big(\frac{I_0}{s}\Big) = -\frac{1}{s^2} I \quad , \tag{26}$$

$$\frac{dI}{ds} = -\frac{1}{2} I_0 + I \quad . \tag{27}$$

Combining (26) and (27), we get a second order differential equation for $I(s)$,

$$sI''(s) - (1+s)I'(s) + \frac{1}{2} I(s) = 0 \quad . \tag{28}$$

The series solution to eq. (28) is given as

$$
\begin{aligned}
I(s) &= c_1 I_1(s) + c_2 I_2(s) \quad , \\
I_1(s) &= s^2 F(s;\frac{3}{2},3) = s^2\Big\{1+\frac{1}{2}s+\frac{5}{32}s^2+\frac{7}{192}s^3+\frac{7}{1024}s^4+\cdots\Big\} \quad , \\
I_2(s) &= I_1(s)\int \frac{se^s}{[I_1(s)]^2}ds \\
&= -\frac{1}{16}s^2\ln s\Big(1+\frac{1}{2}s+\frac{5}{32}s^2+\cdots\Big)-\frac{1}{2}\Big(1+\frac{1}{2}s+\frac{5}{32}s^2+\frac{7}{192}s^3+\frac{7}{1536}s^4+\cdots\Big) \ ,
\end{aligned}
\tag{29}
$$

where $F(s;\frac{3}{2},3)$ is the confluent hypergeometric function which is convergent for any finite s, and the integral constants $c_1 \simeq -0.095$, $c_2 = -1$. See Appendix for the derivation of these numerical values for c_i.

Finally, collecting (21), (22) and (23), the expectation value of H is written as

$$
\begin{aligned}
\langle H\rangle = M &+ \frac{1}{2M}\Big(\frac{3}{2}\mu^2\Big)+\frac{2}{\sqrt{\pi}}(-\alpha_c\mu+K/\mu) \\
&+\frac{2\mu}{\sqrt{\pi}}\Big[1+\frac{1}{2}(m/\mu)^2+\Big(\frac{5}{32}-2c_1\Big)(m/\mu)^4+\frac{1}{4}(m/\mu)^4\ln(m/\mu)\Big] \quad ,
\end{aligned}
\tag{30}
$$

up to $(m/\mu)^4$.

With the input value of $m = m_{sp} = 0.15$ GeV, we minimize $\langle H\rangle$ of (30), and we obtain

$$
\begin{aligned}
p_F = \bar{\mu} &= 0.54\ GeV, & m_B = \bar{E} &= 5.54\ GeV & \text{for } \alpha_s &= 0.35 \quad , \\
\bar{\mu} &= 0.49\ GeV, & \bar{E} &= 5.63\ GeV & \text{for } \alpha_s &= 0.24 \quad .
\end{aligned}
\tag{31}
$$

For comparison, we calculated $\langle H\rangle$ for the case of $m = 0$ in which the integral of the square root operator is exact, and we get

$$
\begin{aligned}
\bar{\mu} &= 0.53\ GeV, & \bar{E} &= 5.52\ GeV & \text{for } \alpha_s &= 0.35 \quad , \\
\bar{\mu} &= 0.48\ GeV, & \bar{E} &= 5.60\ GeV & \text{for } \alpha_s &= 0.24 \quad .
\end{aligned}
\tag{32}
$$

The calculated values of the B-meson mass, $\bar{E}$, are much larger than the measured value of 5.28. The large values for the mass are originated partly because the

Hamiltonian (30) does not take care of the correct spin dependences for B and B^*. The difference between the pseudoscalar meson and the vector meson is given arise to by the chromomagnetic hyperfine splitting, which is

$$V_s = \frac{2}{3Mm}\,\vec{s}_1 \cdot \vec{s}_2\,\nabla^2(-\frac{\alpha_c}{r}) \quad . \tag{33}$$

Then the expectation value of V_s is given as

$$\begin{aligned} \langle V_s \rangle &= -\frac{2}{\sqrt{\pi}}\frac{\alpha_c \mu^3}{Mm} \qquad \text{for} \quad B \quad , & (34-1)\\ &= \frac{2}{3\sqrt{\pi}}\frac{\alpha_c \mu^3}{Mm} \qquad \text{for} \quad B^* \quad , & (34-2) \end{aligned}$$

and we treat $\langle V_s \rangle$ only as a perturbation. And we get for B meson

$$\begin{aligned} p_{_F} &= 0.54\ GeV, \qquad \bar{E}_B = 5.42\ GeV \qquad \text{for}\ \alpha_s = 0.35 \quad , \\ p_{_F} &= 0.49\ GeV, \qquad \bar{E}_B = 5.56\ GeV \qquad \text{for}\ \alpha_s = 0.24 \quad . \end{aligned} \tag{35}$$

The perturbative result for B^* is

$$\begin{aligned} p_{_F} &= 0.54\ GeV, \qquad \bar{E}_{B^*} = 5.58\ GeV \qquad \text{for}\ \alpha_s = 0.35 \quad , \\ p_{_F} &= 0.49\ GeV, \qquad \bar{E}_{B^*} = 5.65\ GeV \qquad \text{for}\ \alpha_s = 0.24 \quad . \end{aligned} \tag{36}$$

The calculated values of the B-meson mass, 5.42GeV ($\alpha_s = 0.35$) and 5.56GeV ($\alpha_s = 0.24$) are in reasonable agreement compared to the experimental value of $m_B = 5.28$GeV; the relative errors are 2.7% and 5.3%, respectively. But for Fermi motion parameter $p_{_F}$, the calculated values, 0.54GeV ($\alpha_s = 0.35$) and 0.49GeV ($\alpha_s = 0.24$), are somewhat larger than the value 0.3, widely used in the experimental analyses of energy spectrum of semileptonic B meson decay. The value $p_{_F} = 0.3$ corresponds to the B-meson radius $R_B \sim 0.66$ fm, and it seems too large to us. On the other hand, the value $p_{_F} = 0.5$ corresponds to $R_B \sim 0.39$ fm, which looks in reasonable range.

The Fermi motion parameter $p_{_F}$ is the most important parameter of ACCMM model, and the value $p_{_F} \sim 0.3$ GeV has been widely used in experimental analyses without clear theoretical or experimental evidence. Therefore, it is strongly required to determine the value of $p_{_F}$ more firmly when we think of the importance of its role in experimental analyses. We attempted to extract the possible value for $p_{_F}$ theoretically in the relativistic quark model using with quantum mechanical variational method. The better determination of $p_{_F}$ is also interesting theoretically since it has the physical correspondence related to the Fermi motion inside B-meson. We obtained $p_{_F} \sim 0.5$ GeV, which is somewhat larger than 0.3, and we also derived the ground state eigenvalue of $E_B \simeq 5.5$ GeV, which is is reasonable agreement with $m_B = 5.28$ GeV.

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Appendix

The integration constants c_1 and c_2 in (29) are given by the following relations,

$$I(0) = -\frac{1}{2}c_2 = \int_0^\infty x^3 e^{-x^2} dx = \frac{1}{2} \quad , \qquad (A-1)$$

$$\begin{aligned} I''(s \approx 0) &= 2c_1 + c_2(-\frac{1}{8}\ln s - \frac{11}{32}) \\ &= -\frac{1}{4}\int_0^\infty x^2(x^2+s)^{-3/2} e^{-x^2} dx \quad \text{at} \quad s \approx 0 \quad . \end{aligned} \qquad (A-2)$$

Then, from (A-1), we get

$$c_2 \quad = \quad -1 \quad . \qquad (A-3)$$

The integral in (A-2) can be expanded as

$$\begin{aligned} J(s=a^2) &= \int_0^\infty x^2(x^2+a^2)^{-3/2}e^{-x^2}dx \\ &= \int_0^\infty x^2\left[(x+a)^2-2ax\right]^{-3/2}e^{-x^2}dx \\ &= \int_0^\infty x^2(x+a)^{-3}\left[1-\frac{2ax}{(x+a)^2}\right]^{-3/2}e^{-x^2}dx \\ &= \sum_{n=0}^\infty \frac{(2n+1)!a^n}{2^n(n!)^2}\int_0^\infty \frac{x^{n+2}}{(x+a)^{2n+3}}e^{-x^2}dx \quad . \qquad (A-4) \end{aligned}$$

Next the integral in (A-4) is obtained by

$$\int_0^\infty \frac{x^{n+2}}{(x+a)^{2n+3}}e^{-x^2}dx = \frac{1}{(2n+2)!}\left(\frac{\partial}{\partial a}\right)^{2n+2}\int_0^\infty \frac{x^{n+2}}{x+a}e^{-x^2}dx \quad . \qquad (A-5)$$

Again the integral in (A-5) is related to another integral, for a small value of a,

$$\int_0^\infty \frac{x^{n+2}}{x+a}e^{-x^2}dx = \sum_{k=0}^{n+1}\frac{(-a)^k}{2}\left(\frac{n-k}{2}\right)! + (-a)^{n+2}\int_0^\infty \frac{e^{-x^2}}{x+a}dx \quad . \qquad (A-6)$$

The integral in (A-6) can be expanded in a similar way as to obtain the series (29) by making use of differential eqations. For a small value of a,

$$\int_0^\infty \frac{e^{-x^2}}{x+a}dx = -\frac{1}{2}e^{-a^2}(2\ln a+\gamma+a^2+\frac{1}{2}\frac{a^4}{2!}+\cdots)+\sqrt{\pi}e^{-a^2}(a+\frac{1}{3}a^3+\frac{1}{5}\frac{a^5}{2!}+\cdots) \quad , \qquad (A-7)$$

where $\gamma \sim 0.5772$ is the Euler's constant. In this way the constant c_1 is given by

an infinite series,

$$c_1 = -\frac{3}{64} + \frac{\gamma}{16} - \frac{1}{8}\sum_{n=1}^{\infty}\frac{1}{n2^n} \approx -0.0975 \quad . \qquad (A-8)$$

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